Internship Defense

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Plan of the presentation

Treewidth

- Tree decomposition :
  - A tree $\mathscr{T}$ where each node corresponds to a subset of nodes of the original graph
  - Each vertex and each edge is in at least one node of $\mathscr{T}$
  - The set of nodes containing a vertex v form a connected component of $\mathscr{T}$
- We seek to minimize the size of the biggest set in $\mathscr{T}$:
  - That number minus 1 is defined as the treewidth.
  - It is $\mathbf{NP-complete}$ to determine the treewidth of a graph.
- Many $\mathbf{NP-hard}$ problems can be polynomially parameterized by treewidth :
  - Coloring, hamiltonicity, weighted independent set
  - Courcelle's Theorem (1990)

Grid Minor

Definition

- Grid : *n* horizontal and vertical lines with nodes at the junction
- Minor : subgraph obtaining by deleting and contracting edges from the original graph

Theorem

Every graph with treewidth k has a grid minor of size $\Omega(k^{1/98+o(1)})$

Proof Scheme

1. We start with a graph G with treewidth k
2. Get a bounded-degree minor G' with treewidth $\Omega(k/\text{polylog} k)$
3. Get a clustering with good routers and paths between them
4. Make a tree-of-set from the clusters
5. Get a path-of-set from the tree-of-set
6. Get a grid minor from the path-of-set

Initial Simplifications

With only polylog losses in the treewidth k , we can :

- Get a graph G' with bounded degree
- Get a subset of $\Omega(k/\text{polylog}k)$ nodes of G' that is node-well-linked, that is :

Definition

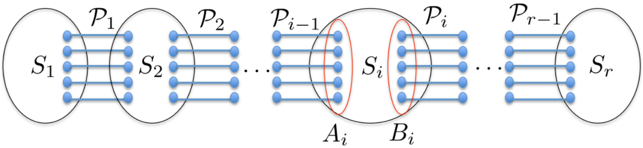
A set is node-well-linked if for any pair (T_1, T_2) of equal sized subsets we can join the pair by $|T_1|$ node-disjoint paths.

Path-of-set

Definition

A path-of-set of width r and height h consists of :

- A sequence $S = \{S_1, ..., S_r\}$ or r disjoint subgraphs of g where each $G[S_i]$ is connected ;
- For $1 \leq i < r$ a set P_i of h disjoint paths connecting S_i to S_{i+1} without using any of the other vertices present in the S_j, such that all paths in $\bigcup_i P_i$ are mutually disjoint ;

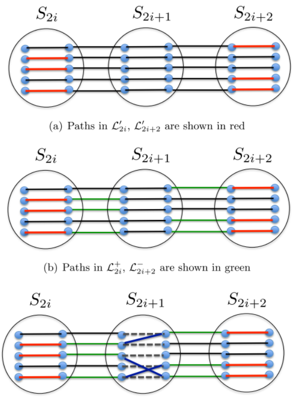


Path-of-set to Grid Minor

Theorem

We can find a grid minor of size $\sqrt{h} \times \sqrt{h}$:

- We can always find a set of $\Omega(\sqrt{h})$ horizontal paths where we can add any vertical edge.
- We use the routers to connect $h/2$ such sets and add a vertical edge in each.

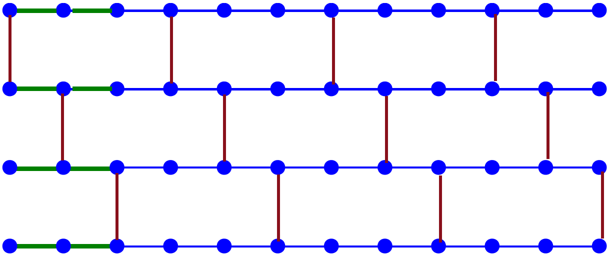


(a) Paths in $\mathcal{L}'_{2i}$, $\mathcal{L}'_{2i+2}$ are shown in red

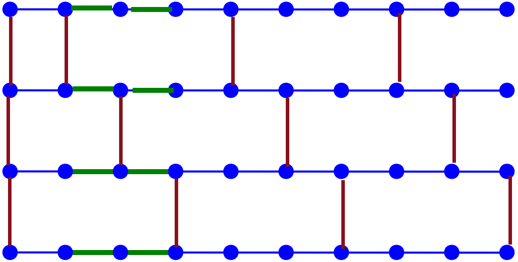
(b) Paths in $\mathcal{L}^{+}_{2i}$, $\mathcal{L}^{-}_{2i+2}$ are shown in green

(c) Paths in $\mathcal{R}_{2i+1}$ are shown in purple.

Getting the Grid



We contract the green edges to get



Tree-of-set

Definition

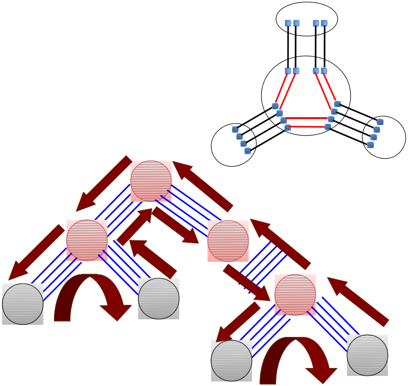
A tree-of-set of width of width h consists of :

- A collection $\mathscr{S} = \{S_1, \ldots, S_r\}$ of r disjoint vertex subsets of G, where each $G[S_i]$ is connected ;
- A binary tree T over a set $\{v_1, \ldots, v_r\}$ of vertices ;
- For each edge $e = (v_i, v_j)$ in T a set P_e of h disjoint paths connecting S_i to S_j that doesn't use any vertice from other S_k with all paths in $\bigcup_{e \in E(T)} P_e$ pairwise disjoint.

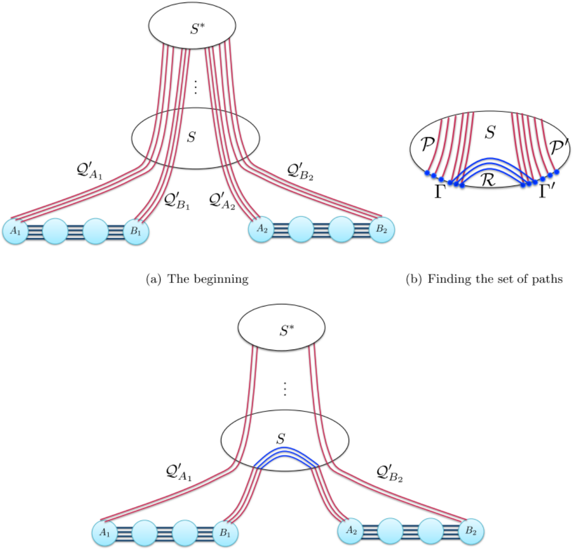
We are looking for a path-of-set inside the tree-of-set.

Tree-of-set to Path-of-set

- Basic idea : do a DFS through the tree.
- Simply going through every node will not work
- Use the leaves as clusters for the path-of-set



Routing in the Tree-of-set



(a) The beginning

(b) Finding the set of paths

(c) The end

Initial Clustering

- Start with node-well-linked set
- Add virtual *terminals*
- Keep $\mathscr{T}$ node-well-linked while removing edges until G' is minimal
- Find a clustering that minimizes the number of edges between clusters

Clustering the graph

- Good routers :
  - No terminals inside
  - High bandwidth
  - Can send lots of flow to the terminals
- Good clustering :
  - All clusters are small
  - Each terminal is its own cluster
  - All clusters have high bandwidth
- We replace big clusters by smaller ones and alternate clusterings with high bandwidth and clusterings with smaller clusters each time until we obtain what we want

Improving the bound

- The bound was improved multiple times
- Some properties are not yet used which could give better results (e.g. bounded degree)
- Some steps seem polynomially tight : path-of-set to grid for example
- So far existential approach (non algorithmic) only gave polylog gains
- The clusters to tree-of-set seems the most promising area of improvement