# Dynamic Sum-Radii Clustering

## Talk at Algotel 2016

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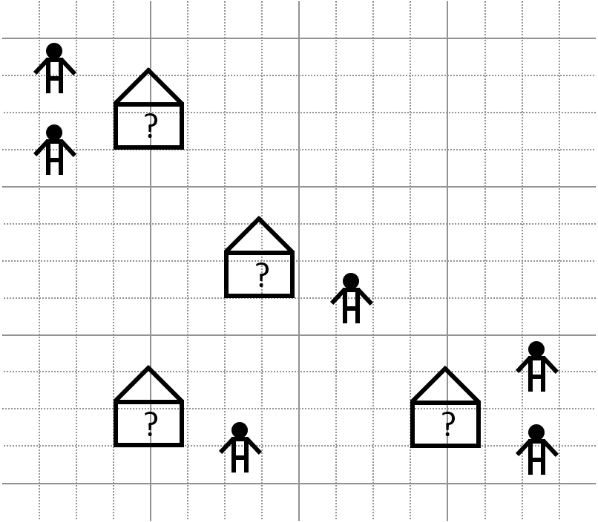
# Plan of the talk

1. Presentation of the problem
   - Facility Location
   - Reduction from SET-COVER
2. Algorithms in the general (non-metric) case
   - Linear Program
   - Preprocessing
   - $\log(nT)$-Approximation
   - $\log n$-Approximation
3. Metric case
4. Conclusion

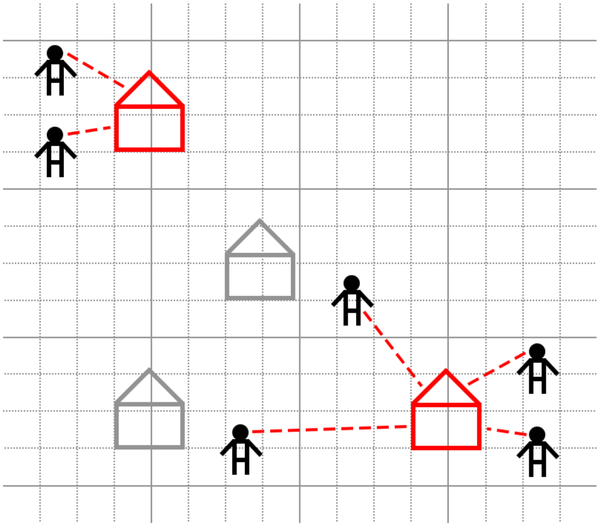
# Presentation of the problem

# Warehouse Location



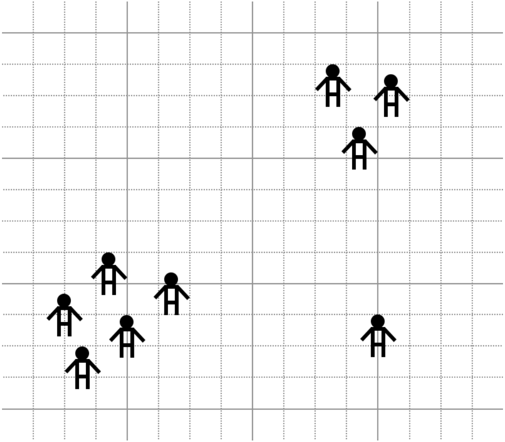

Introduced by Hamburger and Kuehn in 1963

# Warehouse Location



Minimizing opening and connection costs

# Use for Clustering



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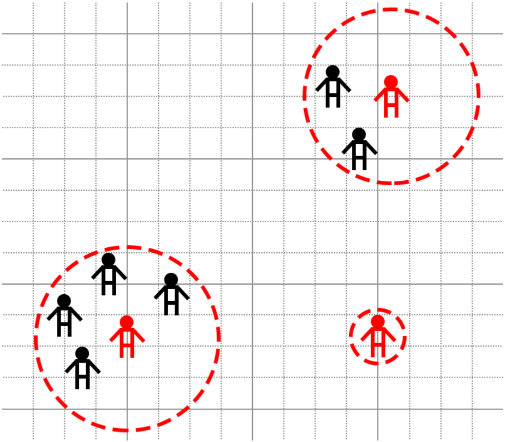

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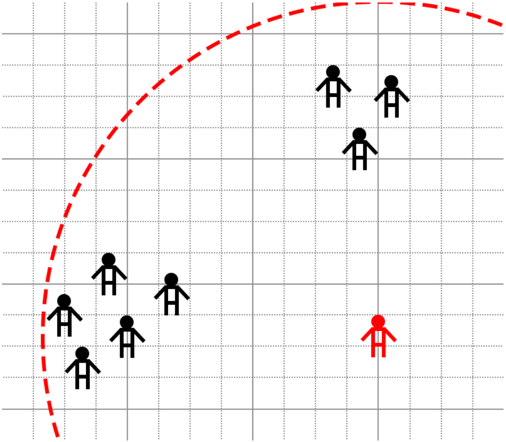
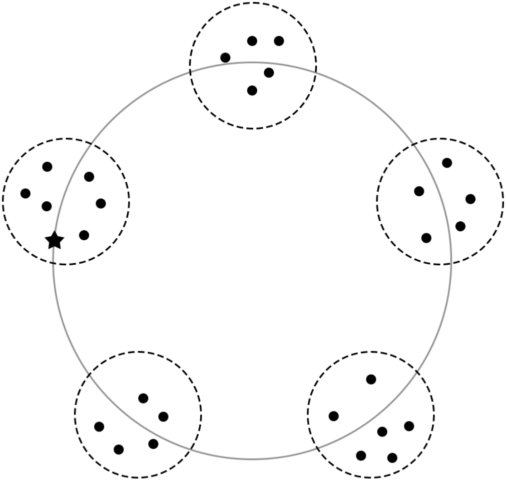
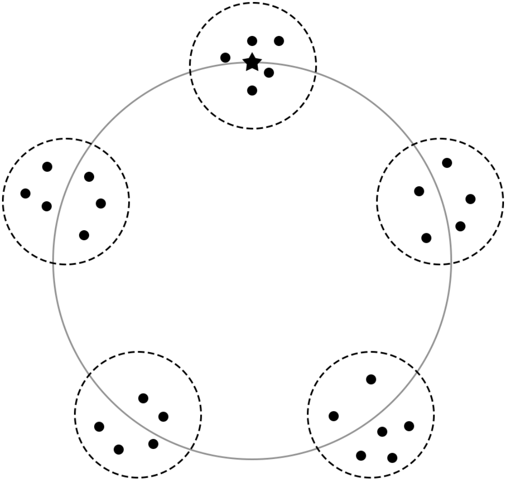


# Dynamic aspect



Dynamic case studied in [EMS 2014] (Introduced by Berman and Odoni in 1982 for stochastic clients)

# Dynamic aspect



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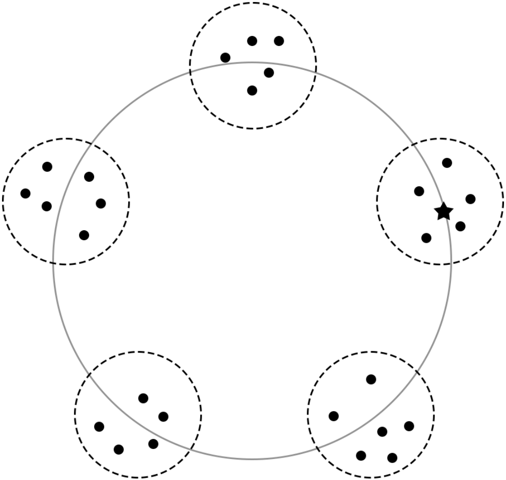

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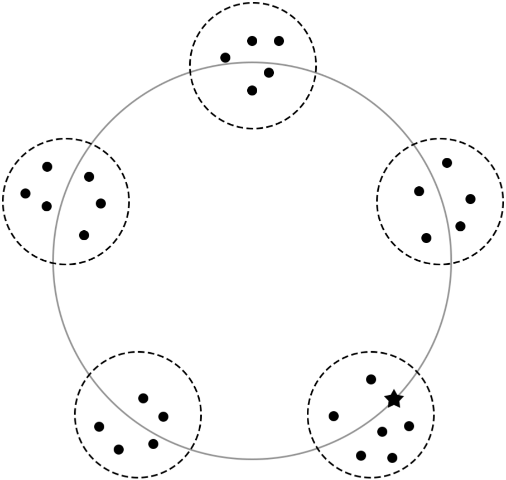

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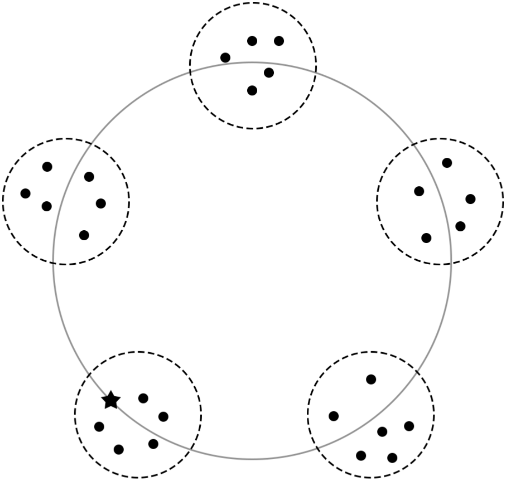

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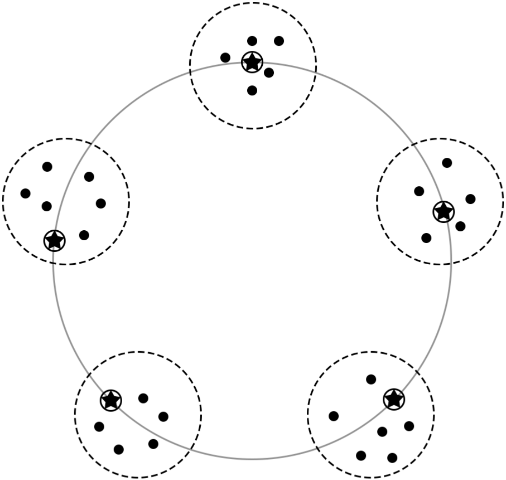

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# Definitions

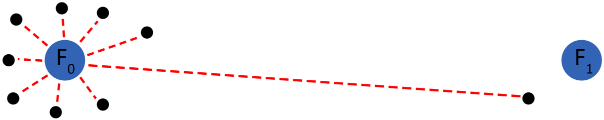
We have as input:

- Set $\mathcal{C}$ of $n$ clients
- Set $\mathcal{F}$ of $m$ facilities
- Set $\mathcal{T}$ of $t$ time steps
- Opening cost $f$ and switching cost $g$
- Distances $d^t(i,j)$ for each (facility, client) pair and each time step

We seek to connect all clients and minimize costs

# Problem with the cost function : Sum of distances

# Problem with the cost function : Sum of distances



# New cost function : Sum of radii


Model studied by Charikar and Panigrahy in 2001 (static case)

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# Reduction from SET-COVER

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**Theorem**

*[Dinur - Steurer, 2013] SET-COVER has no $(1-\varepsilon)\log n$ approximation unless $\mathbf{P} = \mathbf{NP}$.*

Which gives us :

**Theorem**

*Dynamic Facility Location Minimizing Sum of Radii has no $(1-\varepsilon)\log n$ approximation unless $\mathbf{P} = \mathbf{NP}$.*

However this result doesn't hold in the metric case.

# Algorithms in the general (non-metric) case

# Formulation en Programme Linéaire

Three sets of boolean variables with :

- $y_{ir}^t$ equals 1 iff $i$ is open with radius $r$ at time $t$
- $x_{ij}^t$ equals 1 iff $j$ is connected to $i$ at time $t$
- $z_{ij}^t$ equals 1 iff $j$ connected to $i$ between $t-1$ and $t$

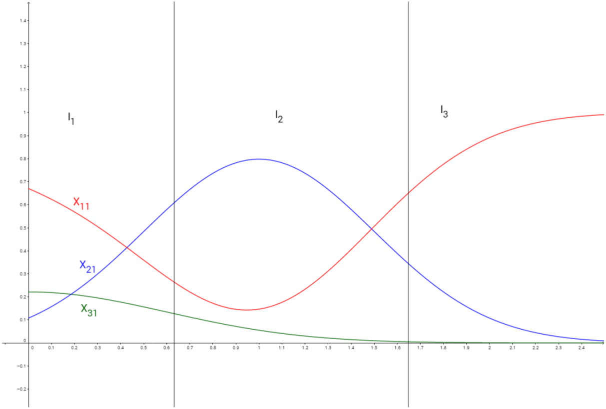
Minimize

$$\sum_{i,t,r} y_{ir}^t \cdot (f+r) \quad + \quad g \sum_{i,j,t} z_{ij}^t$$

With the constraints :

- $\forall j,t \quad \sum_i x_{ij}^t \geq 1$
- $\forall i,j,t \quad x_{ij}^t \leq \sum_{r \geq d^t(i,j)} y_{ir}^t$
- $\forall i,j,t \geq 1 \quad z_{ij}^t \geq x_{ij}^t - x_{ij}^{t-1}$

# Preprocessing : choosing when clients switch facilities




## Lemma

*(EMS) Preprocessing multiplies the cost by at most 2*

## Opening facilities

Algorithm 1 :

- Repeat $\log(Z) + 1$ times
  - For each facility $i$, draw $a_i$ uniformly in $[0, 1]$
  - For each time step $t$, open facility $i$ with the biggest radius $R$ such that

$$a_i \leq \sum_{r \geq R} y_{ir}^t$$

    *It's equivalent to opening i with a radius distributed along the $y_{ir}^t$, while maintaining timewise coherence*

- Open each facility with the biggest radius among the $\log(Z) + 1$ partial solutions

# Covering each client

Interval $I$ of client $j$ isn't covered by facility $i$ if there is $t \in I$ such that

$$a_i \geq \sum_{r \geq d^t(i,j)} y_{ir}^t$$

By the LP's second constraint, for $t \in I$

$$\sum_{r \geq d^t(i,j)} y_{ir}^t \geq \min_{t \in I} x_{ij}^t \stackrel{\text{def}}{=} x_{ij}^I$$

The probability of being covered is then at least

$$1 - \left( \prod_i \left(1 - x_{ij}^I\right) \right) \underset{\text{LP}}{\geq} \frac{1}{2}$$

Repeating $\log(Z) + 1$ times, each client-interval is covered with probability at least $\frac{1}{2}$

# Total Cost

- Switching costs at most $Z \cdot g$, by preprocessing at most $2 \cdot OPT$
- Cost expectation by facility equal to $\sum y_{ir}^t (r + f)$
- Expecation of cost for partial solutions equal to $OPT$
- Recombining costs less than summing

## Theorem

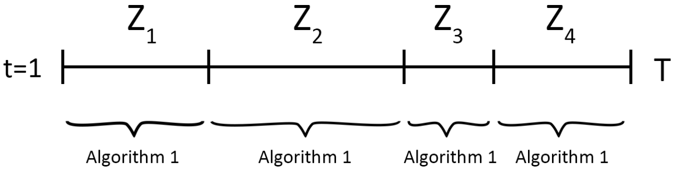
*Algorithm 1 is an $O(\log(Z))$ approximation*

## Corollary

*Algorithm 1 is an $O(\log(nT))$ approximation*

# $\log n$-Approximation

- If $Z \leq n^2$ use Algorithme 1
- Else cut time in sequences such that $n \leq Z_i < 2n$ and use 'Algorithm 1 on each sequence



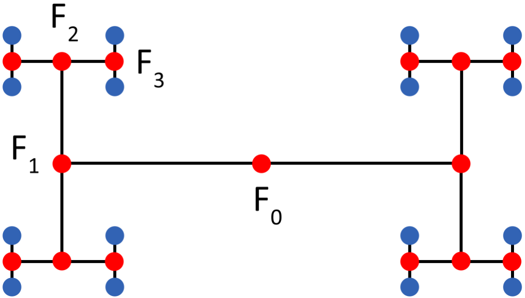

## Theorem

*Algorithm 2 is an $O(\log n)$ approximation*

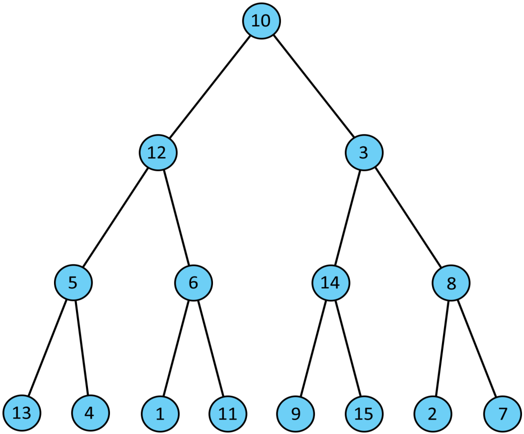
# Metric case

# ANS Algorithm

- There is an $O(1)$ approximation for the sum of distances [An, Norouzi-Fard, Svensson, 2014]
- We propose a natural adaptation of this algorithm
- It can be at best a $\Omega(\log \log n)$ approximation
- Counter-example :



# Combinatorial Lemma



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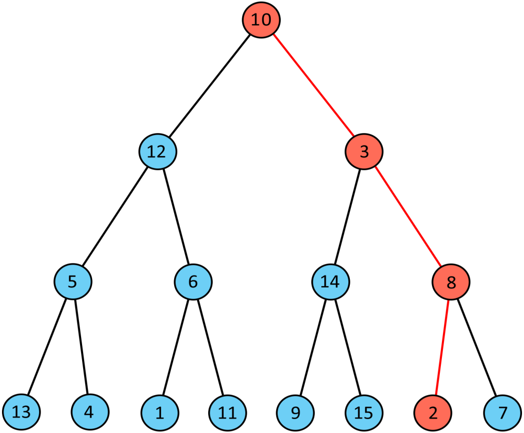
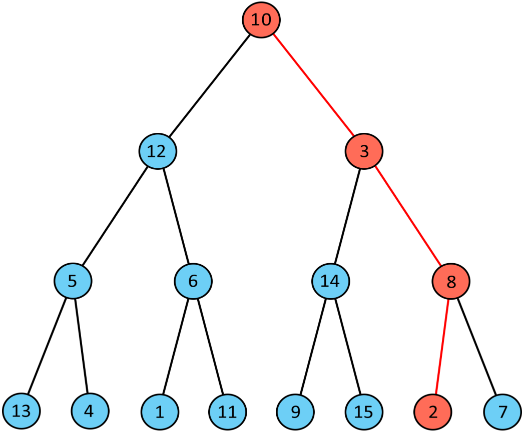


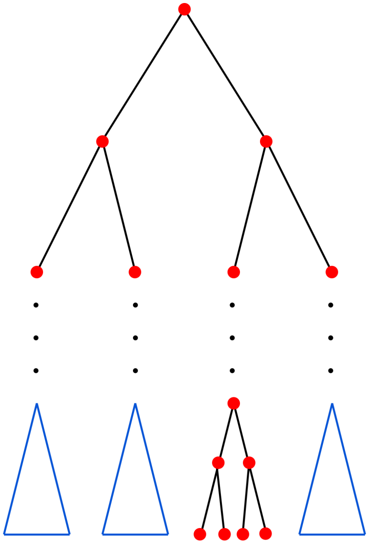
# Combinatorial Lemma




## Lemma

*Finding such a branch has probability at least* $\frac{1}{3}$ .

# Combinatorial Lemma(2)



# Conclusion

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Our results on Sum-Radii Clustering :

| | Non-Metric (Algorithm/Hardness) | |
|---|---|---|
| | $\sum$ Distances | $\sum$ Radii |
| Static | $O(\log n)$ / $\Omega(\log n)$ | $2\log n$ / $(1-\varepsilon)\log n$ |
| Dynamic | $O(\log n\cancel{T})$ / $\Omega(\log n)$ | $4\log n$ / $(1-\varepsilon)\log n$ |

Moreover, modified ANS cannot give better than $\Omega(\log\log n)$ for the sum of radii.

# Future work

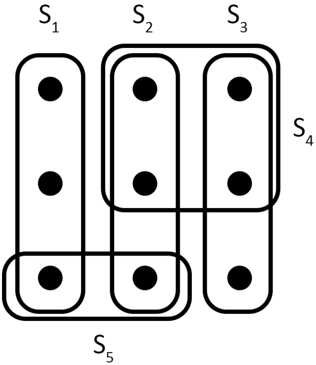
Remaining problems :

- Can we find an upper bound for modified ANS ?
- Can we find a constant approximation algorithm in the metric case ?

Any questions ?

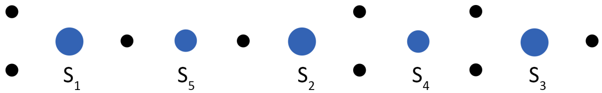
# SET-COVER problem

- Set $\mathcal{A}$ with $n$ elements
- Collection $\mathcal{B}$ of subsets of$\mathcal{A}$
- How to cover$\mathcal{A}$ with a minimal cardinality subset of $\mathcal{B}$ ?



# SET-COVER reduction

- A client $j$ for each element $a_j \in \mathcal{A}$
- A facility $i$ for each set $b_j \in \mathcal{B}$
- Set $d(i,j) = 1$ if $a_j \in b_i$ and $d(i,j) = \infty$ otherwise



# Preprocessing

- Set $t_0 = 1$
- Greedily find the greatest $t_{k+1}$ such that

$$\sum_i \min_{t \in [t_k, t_{k+1}[} x_{ij}^t \geq \frac{1}{2}$$

- Set for all $[t_k, t_{k+1}]$

$$\widehat{x_{ij}^t} \stackrel{\text{def}}{=} 2 \cdot \min_{t \in [t_k, t_{k+1}[} x_{ij}^t$$

- And

$$\widehat{y_{ir}^t} \stackrel{\text{def}}{=} 2 \cdot y_{ir}^t$$

# Proof of the combinatorial lemma

- The root's key is in the top 1/3 with probability 1/3
- If we only take an $\varepsilon$ proportion of nodes:
  - Each node's key has probability at least $\frac{2}{3} - \varepsilon$ to be smaller than the root's.
  - We can find a path through a subtree of height $\frac{n}{2}$ with constant probability
  - From this node at height $\frac{n}{2}$ we can find an other path of length $\frac{n}{2}$ hence a branch with the property
- We handle dependency and the value of $\varepsilon$ by only looking at an arbitrarily small proportion of nodes

## Lemma

*A good branch exists with probability at least*

$$1 - \ln(2) \approx 0.3068$$

# Modified ANS Algorithm

- Take an LP solution
- Use the preprocessing
- Draw $a_i \in \mathbb{R}$ for each facility with distribution along $e^{-a_i x}$
- Draw an uniform permutation of clients
- Draw an arrow from client $j$ to facility $i$ with minimal $a_i$ with $x_{ij}^t > 0$
- Draw an arrow from each facility to the client of lowest rank such that $x_{ij}^t > 0$
- Assign each client to the facility in the loop of length2 you get by following the arrows

# Experimental results

Series of 10 groups of $10^7$ instances for the combinatorial lemma:

| Height | 5 | 6 | 7 |
|---|---|---|---|
| Observed probability | 0.473 | 0.453 | 0.441 |
| Spread over 10 groups | 0.0008 | 0.0018 | 0.0015 |

| Height | 8 | 9 | 10 |
|---|---|---|---|
| Observed probability | 0.431 | 0.423 | 0.418 |
| Spread over 10 groups | 0.0013 | 0.0021 | 0.0017 |