A note on covering balls with minimal radius

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Consider a set of points in a metric space E , and to each point i associate a closed ball of radius r_i (corresponding to the points in E that are at distance $\leq r_i$ from i). We have one constraint on the balls : for any pair (i, j) of points there is a path from i to j that only goes through our set of balls (that is, the subspace S made from the union of balls is path-connected).

Problem. The question we ask is the following : by increasing the radius of one ball i such that it covers every point in S, how can we minimize the ratio of the new radius compared to the sum of the previous radii, hence how do we minimize

$$\frac{r_i'}{\sum_j r_j}$$

The immediate answer is to find the ball closest to the "center" of S and open it with a sufficient radius, but that center might not be easy to manipulate in some abstract metric spaces and does not immediately give the best bound. Before proving the result we need a small lemma.

Lemma. *If a shortest path P between two points x, y in S through the set of balls is of length d, the sum of radii is at least $d/2$.*

Proof. The length of the path inside a ball j is at most $2r_j$ (because the path is as short as possible and any point inside the ball centered on j is at distance at most $2r_j$ from any other). Hence the sum over all the balls the path crosses of their radius is at least $\frac{d(y,x)}{2}$. □

Remark. This means that we can easily have $\frac{r_i'}{\sum_j r_j} \leq 2$, by opening any facility with a big enough radius to cover everyone. Suppose that we take a facility i and a point x farthest from i, opening i with radius $d(i, x)$ is enough to cover S (because a point not inside that new ball would have to be at distance higher than $d(i, x)$). But the cost of the initial solution was at least $d(i, x)/2$, hence the result.

Corollary. *The previous result can be extended to arbitrary paths as long as their total length inside each ball is at most $2r_i$.*

Our goal is now to prove the following :

Theorem. *We can always cover S with $r_i' \leq r + \sum_j r_j$, where r is the mean of the two biggest radii among the r_i.*

Proof. Consider a graph where each i is a vertex, and we have an edge (i, j) iff $d(i, j) \leq r_i + r_j$ (that is, the balls intersect) and set a weight on each edge equal to $d(i, j)$. Let's now consider a minimum spanning tree T of that graph. For each node i of T we also add a leaf corresponding to each point of i furthest from each of the center of the balls in S, and udpate the distance accordingly (this results in an increase of at most r_i and a number of new leaves at most equal to the square of the number of nodes). We obtain another tree T'. If we open the correct ball with the radius of the tree we know that we cover all of S, because for each center i, the furthest point from i is in one of the balls of T, hence was created in T', and is taken into account when finding the radius. And we know that the sum of the radii is at least half of the diameter of the graph. However, we know that $2 \times radius - \max(d(i, j)) \leq diameter$[1]. So we already know that the difference between the final radius and the sum of radii is $\max(d(i, j))/2$, but $d(i, j)$ is at most the sum of the two biggest radii, hence the cost difference is at most r. □

Corollary. *This bound is tight.*

Proof. We can see it first in the case of two balls of radius r touching in a single point. One of them has to be opened with radius $3r$ to cover the whole subspace, hence a tight bound. It is also true if we have an even number $2n$ of identical balls of radius r in a line, plus a ball of high radii r' on each side of the middle of the line. The best solution is to open one the two big balls with sufficient radius. The total cost in this case is $2n \times r + 3r'$, and the initial cost was $2n \times r + 2r'$. □

Remark. If we only consider a multiplicative bound, this means that we can get examples with $\frac{3}{2} \leq \frac{r_i'}{\sum_j r_j}$ (as in the first previous example), and the theorem becomes $\frac{r_i'}{\sum_j r_j} \leq \frac{3}{2}$.

[1] https://www.csie.ntu.edu.tw/~kmchao/tree07spr/diameter.pdf